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Estimates of Entropy Numbers and Gaussian Measures for Classes of Functions with Bounded Mixed Derivative

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The paper contains estimates for the entropy numbers of classes of functions with conditions on the mixed derivative (difference), in the uniform and integral metrics. As an application, the new estimates of the Gaussian measure of a small ball are obtained.

INTRODUCTION

For classes of periodic functions of d real variables we obtain estimates of ε-entropy in the uniform and integral metrics. These results are compared with the ε-entropy estimates in the metric of a special Besov space which appears naturally in many approximation problems. Combined with recent results of Kuelbs and Li, these estimates yield new estimates of the Gaussian measure of a small ball.

Let us recall the definitions (cf. [29]). Let K be a compact set in the Banach space X. The ε-entropy $\mathscr{H}_\varepsilon(K; X)$ (or simply $\mathscr{H}(K, \varepsilon)$) is the logarithm to the base two of the number of points in the minimal ε-net. We use also the inverse quantities, the so-called entropy numbers, given by

$$\varepsilon_m(K; X) = \inf \left\{ \varepsilon \colon K \subset \bigcup_{j=1}^{2^m} (x_j + \varepsilon B_X) \right\}.$$

The infimum is taken over all ε such that K can be covered by 2^m balls εB_X of radius ε.

The main properties of the ε-entropy or entropy numbers can be found, for example, in [29, 19]. We give some of them below.

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Let $\mathbb{R}^d$ be the d-dimensional Euclidean space, $Q^d = [0, 1)^d$ the unit cube and $\mathbf{r} = (r_1, ..., r_d)$ a vector with nonzero coordinates. For the sake of convenience we suppose that the coordinates are ordered as follows $0 < r_1 = r_2 = \cdots = r_\nu < r_{\nu+1} \leqslant \cdots \leqslant r_d$.

Let $W_p^{\mathbf{r}}$ be the class of periodic functions f defined on Q^d such that the norm of the mixed derivative of order $\mathbf{r}$ is bounded, that is, $\|f^{(r_1, ..., r_d)}\|_p \leqslant 1$, and $\int_0^1 f(x_1, ..., x_d)\, dx_j = 0$, for $j = 1, ..., d$. For r_j fractional the derivative is understood in the Weyl sense. A function of the class $W_p^{\mathbf{r}}$ has the following integral representation

$$f(\mathbf{x}) = \int_{Q^d} \phi(\mathbf{x} - \mathbf{t})\, K_{\mathbf{r}}(\mathbf{t})\, d\mathbf{t},$$

where $\|\phi\|_p \leqslant 1$ and $K_{\mathbf{r}}(\mathbf{t})$ is the Bernoulli kernel

$$K_{\mathbf{r}}(\mathbf{t}) = \prod_{j=1}^{d} K_{r_j}(t_j), \qquad K_{r_j}(t_j) = \sum_{k=1}^{\infty} \frac{\cos(2\pi(kt_j + r_j))}{k^{r_j}}.$$

The class $H_p^{\mathbf{r}}$ is the set of functions f such that $\int_0^1 f(x_1, ..., x_d)\, dx_j = 0$ for $j = 1, ..., d$ and for all subsets $\{j_1, ..., j_k\} \subset \{1, ..., d\}$ we have $\|\Delta_{h_{j_1}, ...h_{j_k}}^{l, ..., l} f\|_p \leqslant \prod_{j_s} |h_{j_s}|^{r_{j_s}}$ where the mixed difference of integer order $l > r_d$ is taken with the step h_{j_s} in the variable x_{j_s}.

Such classes are really multidimensional by methodology as well as by results. The first publications on this topic are due to K. Babenko [1, 2]. Many interesting results have been obtained since then but many important problems are still open. A detailed history can be found in the book [26].

This paper gives estimates from above for the entropy numbers on the classes $W_p^{\mathbf{r}}$ and $H_p^{\mathbf{r}}$ in the uniform metric $C(Q^d)$. These estimates are apparently exact since the gap between these estimates and those from below is of order $\sqrt{\log m}$, and in the two-dimensional case the exactness has been already proved (see the work by M. Talagrand [24], and also [28]). The order of entropy numbers in the integral metric $L_p(Q^d)$, $1 < p < \infty$, is found for all $\mathbf{r}$ for which the embedding theorems hold. Only estimates from above are proved in the paper. Estimates from below and also estimates from above for $r > 1$ have been known. In the Appendix we answer a question of Kuelbs and Li on direct estimation of certain entropy numbers.

We write $a_m \ll b_m$ if there exists an absolute constant C such that $a_m \leqslant C b_m$, and $a_m \simeq b_m$ if simultaneously $a_m \ll b_m$ and $b_m \ll a_m$.

The plan of the paper is as follows. In Section 1 we formulate the main results, their applications are discussed in Section 2; and proofs are given in Section 3.

1. RESULTS

Our proofs are based mainly on estimates for the entropy numbers in finite-dimensional spaces. Surprisingly some of these estimates that are too crude for the classical Sobolev spaces are sufficient for the classes of functions considered here (at least in the two-dimensional case).

We start with the class $W_p^{\mathbf{r}}$.

THEOREM 1.1. *Let* $r_1 > \max(1/p, 1/2)$, *and* $1 < p < \infty$. *Then*

$$\left(\frac{\log^{\nu-1} m}{m}\right)^{r_1} \ll \varepsilon_m(W_p^{\mathbf{r}}; L_\infty) \ll \left(\frac{\log^{\nu-1} m}{m}\right)^{r_1} \log^{1/2} m.$$

The estimate from above was proved in [5] by a method different from that presented here. Unfortunately, this paper is almost inaccessible (as are some other volumes of Proceedings of Yaroslavl University, edited by Y. Brudnyi). Less precise estimates are given in [28]. The following stronger estimate from below is known

$$\left(\frac{\log^{\nu-1} m}{m}\right)^{r_1} \ll \varepsilon_m(W_\infty^{\mathbf{r}}; L_1).$$

It was proved by the author [5] for $r_1 > 1/2$ (see also [25]), and then by B. Kashin and V. Temlyakov [12] for every $r_1 > 0$ by a more complicated method.

Let us proceed to the classes $H_p^{\mathbf{r}}$.

THEOREM 1.2. *Let* $1 < p < \infty$, *and* $r_1 > \max(1/2, 1/p)$. *Then*

$$\left(\frac{\log^{\nu-1} m}{m}\right)^{r_1} \log^{(\nu-1)/2} m \ll \varepsilon_m(H_p^{\mathbf{r}}; L_\infty) \ll \left(\frac{\log^{\nu-1} m}{m}\right)^{r_1} \log^{\nu/2} m.$$

The estimate from above was proved in [5] for $r_1 > 1$. The estimate from below was proved in [27]. Another proof was given in [5]. Actually, a stronger estimate from below for $r_1 > 0$ is proved in [27] (see also [25]), namely,

$$\left(\frac{\log^{\nu-1} m}{m}\right)^{r_1} \log^{(\nu-1)/2} m \ll \varepsilon_m(H_\infty^{\mathbf{r}}; L_1).$$

Remark 1.3. One notes the gap of $\sqrt{\log m}$ between the estimates from above and below. It seems that the estimates for the space L_∞ which have to be improved are those from below. This has proved to be so in the two-dimensional case [24, 28].

For the case of the integral metric the following exact order of entropy numbers is proved for every $\mathbf{r}$ for which the embedding theorems hold.

THEOREM 1.4. *Let* $r_1 > 1/p - 1/q$, *and* $1 < p \leqslant q < \infty$. *Then*

$$\varepsilon_m(W_p^{\mathbf{r}}; L_q) \simeq \left(\frac{\log^{\nu-1} m}{m}\right)^{r_1}.$$

THEOREM 1.5. *Let* $r_1 > 1/p - 1/q$, *and* $1 < p \leqslant q < \infty$. *Then*

$$\varepsilon_m(H_p^{\mathbf{r}}; L_q) \simeq \left(\frac{\log^{\nu-1} m}{m}\right)^{r_1} \log^{(\nu-1)/2} m.$$

These results were proved for $r_1 > 1$ by methods different than those employed here in [5, 27] (see also [25]), independently. The new result here is the estimate from above for $r_1 > 1/p - 1/q$.

In the one-dimensional case approximation in the C-metric and the stronger metric of the Besov space $B^0_{\infty,1}$ is essentially the same which allows us to often use the $B^0_{\infty,1}$-metrics instead of the C-metrics. Attempts to extend this idea to the multidimensional case can lead to inexact results. We demonstrate this by giving estimates for the entropy numbers in the norm of the Besov space proved in [5]. The estimates of the entropy numbers in the wide range of Besov space metrics were proved in [27]. These results are interesting because, in contrast to the one-dimensional case, the behavior of entropy numbers is different, and they also imply estimates of Gaussian measures.

We use the following definition of the norm in the Besov space $B^0_{\infty,1}$ (cf. [26]),

$$\|f\|_{B^0_{\infty,1}} = \sum_{\mathbf{k} \in \mathbb{Z}^d_+} \|\sigma_{\mathbf{k}} * f\|_\infty,$$

where $\sigma_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^d (V_{2^{k_j+1}}(x_j) - V_{2^{k_j}}(x_j))$ is the product of the one-dimensional de la Vallée Poussin kernel.

THEOREM 1.6. *Let* $r_1 > 1/2$. *Then*

$$\varepsilon_m(W_2^{\mathbf{r}}; B^0_{\infty,1}) \simeq \left(\frac{\log^{\nu-1} m}{m}\right)^{r_1} \log^{(\nu-1)/2} m.$$

THEOREM 1.7. *Let* $r_1 > 1/2$. *Then*

$$\varepsilon_m(H_2^{\mathbf{r}}; B^0_{\infty,1}) \simeq \left(\frac{\log^{\nu-1} m}{m}\right)^{r_1} \log^{\nu-1} m.$$

2. APPLICATIONS

In this section relations between ε-entropy and the Gaussian measure, established by Kuelbs and Li, are used to obtain new estimates for the Gaussian measure.

Let us recall some definitions (see, for example, [14]). Consider the operator $U_{\mathbf{r}}$, $r_1 > 1/2$ from $L_2(Q^d)$ to $C(Q^d)$ given by $U_{\mathbf{r}} f = f * K_{\mathbf{r}}$. If $\{g_k\}$ denotes an independent sequence of standard gaussian random variables then for any given complete orthonormal system $\{f_k\}$ of $L_2(Q^d)$ the series $\sum_k g_k U_{\mathbf{r}} f_k$ converges almost surely in $C(Q^d)$ with the distribution function μ. So, the operator $U_{\mathbf{r}}$ defines the Gaussian measure μ on $C(Q^d)$ which is the image under $U_{\mathbf{r}}$ of the canonical cylindrical Gaussian measure on $L_2(Q^d)$. Let us mention that the space H_μ (the reproducing kernel of μ) is the image of $L_2(Q^d)$ under $U_{\mathbf{r}}$, and its unit ball B_μ is the image under $U_{\mathbf{r}}$ of the unit ball of $L_2(Q^d)$.

It is known that the injection $H_\mu \to C(Q^d)$ is compact. Let us set

$$\phi(\varepsilon) = -\log \mu(x\colon \|x\|_C \leqslant \varepsilon).$$

THEOREM [15]. *The following estimates hold*

$$\phi(2\varepsilon) - \log 2 \leqslant \mathscr{H}\left(B_\mu, \frac{\sqrt{2}\,\varepsilon}{\sqrt{\phi(\varepsilon)}}\right) \leqslant 2\phi(\varepsilon).$$

EXAMPLE 1. Let $H_\mu = W_2^{\mathbf{r}}$ be the Hilbert space of functions periodic on each variable with $\int_0^1 f(x_1, ..., x_d)\, dx_j = 0$, for $j = 1, ..., d$ endowed with the norm $\|f\|_{H_\mu} = \{\int_{Q^d} |f^{(\mathbf{r})}(\mathbf{x})|^2\, \mathbf{dx}\}^{1/2}$.

Let $r_1 > 1/2$. Then

$$\phi(\varepsilon) \gg \frac{\log^{(v-1)(2r_1/(2r_1-1))}(1/\varepsilon)}{\varepsilon^{2/(2r_1-1)}}.$$

This is a direct corollary of the second estimate in the Kuelbs–Li Theorem and estimates from below of Theorem 1.1.

In the case $\mathbf{r} = (1, 1, ..., 1)$ R. Bass [4] proved the estimate

$$\phi(\varepsilon) \ll \frac{\log^{3(d-1)} 1/\varepsilon}{\varepsilon^2}.$$

The following statement strengthens this estimate.

For each $\delta > 0$ there exists a constant C_δ depending only on δ such that

$$\frac{\log^{2d-2} 1/\varepsilon}{\varepsilon^2} \ll \phi(\varepsilon) \leqslant C_\delta \frac{\log^{2d-1+\delta} 1/\varepsilon}{\varepsilon^2}.$$

Proof. The estimate from below is contained in the previous statement. Let us prove the estimate from above. The estimate of ε-entropy in Theorem 1.1 and the first inequality in the Kuelbs–Li Theorem gives

$$\phi(2\varepsilon) \leqslant \frac{\log^{d-1/2} (\sqrt{\phi(\varepsilon)}/\varepsilon)\, \phi(\varepsilon)}{\varepsilon}.$$

Using Bass' estimate for $\phi(\varepsilon)$ we have

$$\phi(2\varepsilon) \leqslant \frac{\log^{(5/2)\, d-2} (1/\varepsilon)}{\varepsilon^2}.$$

Now we can estimate $\phi(2\varepsilon)$ again using this improved estimate instead of that of Bass. A sufficient number of iterations completes the proof. ∎

Example 2. The Hilbert space H_μ is the space of periodic functions $W_2^{\mathbf{r}}$ with $\|f\|_{H_\mu} = \{\int_{Q^d} |f^{(\mathbf{r})}(\mathbf{x})|^2\, \mathbf{dx}\}^{1/2}$ and Gaussian measure μ considered on the Besov space $B^0_{\infty,1}$. Of course, the distribution function μ and the function ϕ correspond to the metric $B^0_{\infty,1}$.

If $r_1 > 1/2$ then

$$\phi(\varepsilon) \simeq \frac{\log^{(\nu-1)((2r_1+1)/(2r_1-1))} (1/\varepsilon)}{\varepsilon^{2/(2r_1-1)}}.$$

This follows from Theorem 1.6 and the Kuelbs–Li result.

3. PROOFS

The proofs of the estimates from above are based on the following lemmas, some of them are well known, others are proved here.

Let $\|\ \|_X$ denote $\mathbb{R}^n$ endowed with the norm of Banach space X, and let X^* be its dual space. Let B^n and S^{n-1} be the Euclidean unit ball and unit sphere. The average of $\|\ \|_X$ on S^{n-1} is denoted by M_X, i.e.,

$$M_X = \int_{S^{n-1}} \|x\|\, d\sigma(x),$$

where σ is the normalized rotation invariant measure on S^{n-1}.

LEMMA 3.1 ([13], see also [20]). $2^{-m/n} \leqslant \varepsilon_m(B_X; X) \leqslant 2(2^{m/n}-1)^{-1}$.

The following lemma is Sudakov's classical result [23].

LEMMA 3.2. *There exists an absolute constant* C, *such that*

$$\mathscr{H}_\varepsilon(B_X; \mathbb{R}^n) \leqslant Cn \left(\frac{M_{X^*}}{\varepsilon}\right)^2.$$

A dual version of this fact was first proved in [21]; a different simple proof was given by A. Pajor and M. Talagrand [10].

LEMMA 3.3. *There exists an absolute constant* C, *such that*

$$\mathscr{H}_\varepsilon(B^n; X) \leqslant Cn \left(\frac{M_X}{\varepsilon}\right)^2, \qquad \varepsilon \geqslant M_X$$

$$\mathscr{H}_\varepsilon(B^n; X) \leqslant Cn \log \frac{M_X}{\varepsilon}, \qquad \varepsilon \leqslant M_X.$$

Observe that the second estimate can be easily derived from the first one and Lemma 3.1. Indeed

$$\mathscr{H}_\varepsilon(B^n; X) \leqslant \mathscr{H}_{M_X}(B^n; X) + \mathscr{H}_{\varepsilon/M_X}(B_X; X)$$

and we have only to use the preceding estimates.

The estimates of Lemma 3.3 can be rewritten for the entropy numbers as follows

$$\varepsilon_m(B^n; X) \ll \begin{cases} \sqrt{n/m}\, M_X, & m \leqslant n \\ M_X e^{-m/n}, & m > n. \end{cases}$$

We now estimate M_X for a special Banach space X. Let E be a subset of $\mathbb{Z}^d$, of cardinality $|E|$. Let X_q^E be the Banach space of trigonometric polynomials with real coefficients

$$t(E; x) = \sum_{k \in E} c_{\mathbf{k}} e^{2\pi i(\mathbf{k}, \mathbf{x})}$$

endowed with the usual norm of the space $L_q(Q^d)$. We denote by $\deg E$ the largest degree of exponentials $e^{2\pi i(\mathbf{k}, \mathbf{x})}$, $\mathbf{k} \in E$, and $\deg e^{2\pi i(\mathbf{k}, \mathbf{x})} = |k_1| + \cdots + |k_d|$.

LEMMA 3.4.

$$M_q^E \equiv M_{X_q^E} \ll \begin{cases} \sqrt{q}, & 2<q<\infty \\ \sqrt{\log \deg E}, & q=\infty. \end{cases}$$

Proof. Let $q<\infty$. By definition

$$M_q^E = \int_{S^{|E|-1}} \left\| \sum_{k \in E} c_k e^{2\pi i(\mathbf{k},\, \mathbf{x})} \right\|_q d\sigma.$$

Let us consider the integral

$$\int |M_q^E(\varepsilon)|^q \, d\varepsilon = \int \left[\int_{S^{|E|-1}} \left\| \sum_{k \in E} \varepsilon_{\mathbf{k}} c_{\mathbf{k}} e^{2\pi i(\mathbf{k},\, \mathbf{x})} \right\|_q d\sigma \right]^q d\varepsilon$$

$$\leqslant \iint_{S^{|E|-1}} \left\| \sum_{k \in E} \varepsilon_{\mathbf{k}} c_{\mathbf{k}} e^{2\pi i(\mathbf{k},\, \mathbf{x})} \right\|_q^q d\sigma \, d\varepsilon,$$

where ε_k are independent random variables taking only two values, ± 1, with equal probability. By the Khinchin inequality (see, for example, [31, Chap. 5]) the last integral is estimated by $Cq^{q/2}$. Hence there exists a distribution of signs $\bar{\varepsilon}$ such that

$$M_q^E(\bar{\varepsilon}) \ll \sqrt{q}.$$

But since the measure σ is rotation invariant, $M_q^E(\bar{\varepsilon}) = M_q^E$.

The case $q=\infty$ is derived from the case $q<\infty$ and the inequality of different metrics [17]

$$\|t(E, x)\|_\infty \leqslant (\deg E)^{d/q} \, \|t(E, x)\|_q.$$

It is sufficient to take $q = d \log(\deg E)$. ∎

We demonstrate all the details in the course of the proof of Theorem 1.1. For the other theorems we concentrate only on necessary alterations.

Proof of Theorem 1.1. We begin with the case $p=2$. Let $\mathbf{s} \in \mathbb{Z}_+^d$ and

$$\rho(\mathbf{s}) = \{\mathbf{k} \in \mathbb{Z}^d \colon 2^{s_j-1} \leqslant |k_j| < 2^{s_j},\ j=1, ..., d\}.$$

Let us denote

$$\delta_{\mathbf{s}}(x) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} c_{\mathbf{k}} e^{2\pi i(\mathbf{k},\, \mathbf{x})}, \qquad \text{and} \qquad \delta_{\mathbf{s}}(f; x) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} \hat{f}(\mathbf{k}) \, e^{2\pi i(\mathbf{k},\, \mathbf{x})}.$$

Let $B_p(j)$ be the subset of trigonometric polynomials $\sum_{r_1 j \leqslant (\mathbf{s}, \mathbf{r}) < r_1(j+1)} \delta_{\mathbf{s}}(\mathbf{x})$ such that

$$\Big\| \sum_{r_1 j \leqslant (\mathbf{s}, \mathbf{r}) < r_1(j+1)} \delta_{\mathbf{s}}(\mathbf{x}) \Big\|_p \leqslant 1.$$

We take $m = 2^l l^{\nu-1}$. Since for each $f \in W_2^{\mathbf{r}}$

$$\Big\| \sum_{r_1 j \leqslant (\mathbf{s}, \mathbf{r}) < r_1(j+1)} \delta_{\mathbf{s}}(f; \mathbf{x}) \Big\|_2 \ll 2^{-jr_1}.$$

(see [18], also [25]), we have

$$\varepsilon_m(W_2^{\mathbf{r}}; L_\infty) \ll \sum_{j=1}^{l} 2^{-jr_1} \varepsilon_{k_j}(B_2(j); L_\infty) + \sum_{j=l+1}^{\infty} 2^{-jr_1} \varepsilon_{k_j}(B_2(j); L_\infty),$$

where the k_j is chosen so that $\sum k_j \leqslant m$.

For $1 \leqslant j < l$ we take $k_j = 2^{j/2 + l/2} j^{\nu-1}$ and use Lemmas 3.3 and 3.4. We obtain

$$\sum_{j=1}^{l-1} 2^{-jr_1} j^{1/2} \exp 2^{(l-j)/2} \ll l^{1/2} 2^{-lr_1}.$$

For the second sum we take only $\sum_l^{\gamma l}$ where γ is a parameter for which the "tail" is $\leqslant l^{1/2} 2^{-lr_1}$. Let us take $k_j = [2^{l(r_1+1/2) - j(r_1 - 1/2)} l^{\nu-1}] - 1$. Now it suffices to take γ so that all $k_j > 0$. By Lemmas 3.3 and 3.4

$$l^{-(\nu-1)/2} 2^{-l/2(r_1+1/2)} \sum_{j=l}^{\gamma l} j^{\nu/2} 2^{-j/2(r_1-1/2)} \ll l^{1/2} 2^{-lr_1}.$$

The estimate is proved.

Let $1 < p < 2$. Suppose for a moment that Theorem 1.4 is proved. We need a particular case of the multiplication property of entropy numbers (for the complete result see [19, Sect. 12.1.5]). Let us consider the operator $U_{\mathbf{r}}\colon L_p \to L_2$ defined by the convolution $f \to f * K_{\mathbf{r}}$. Then the conjugate operator $U_{\mathbf{r}}^*$ takes L_2 into $L_{p'}$.

LEMMA 3.5. *If $r_1 > 1/p$, $r_1' > 1/p - 1/2$, and $r_1'' > 1/2$, then*

$$\varepsilon_{2m-1}(U_{\mathbf{r}}\colon L_p \to L_\infty) \leqslant \varepsilon_m(U_{\mathbf{r}'}\colon L_p \to L_2)\, \varepsilon_m(U_{\mathbf{r}''}\colon L_2 \to L_\infty),$$

where $\mathbf{r} = \mathbf{r}' + \mathbf{r}''$.

Using the estimate for entropy numbers obtained for the case $p = 2$ and Theorem 1.4 (assumed to be true) complete the proof of Theorem 1.1. ∎

Proof of Theorem 1.4. We prove the theorem in a few steps.

Step 1. By Hölder's inequality

$$\|f-g\|_q \leqslant \|f-g\|_2^{1-\theta}\, \|f-g\|_t^{\theta} \leqslant 2\, \|f-g\|_t^{\theta},$$

for every $t > q$, and $1/q = (1-\theta)/2 + \theta/t$. Hence

$$\varepsilon_m(B^n, L_q) \leqslant 2(\varepsilon_m(B^n, L_t))^{\theta} \leqslant 2\left(\sqrt{\frac{n}{m}}\, M_t^E\right)^{\theta}.$$

It is easy to see that t can be chosen so that $\theta/2 > 1/2 - 1/q$ is close to $1/2 - 1/q$. Now the method used in the proof of Theorem 1.1, with Lemma 3.3 replaced by this inequality, proves Theorem 1.4 for $p = 2$, $p < q < \infty$, $r_1 > 1/2 - 1/q$.

Step 2. Let $2 < p < q < \infty$. For every $\lambda > 0$ a trigonometric polynomial $T(x) \in B_p(j)$ can be decomposed into the sum of two polynomials $T(x) = T_1(x) + T_2(x)$ such that $\|T_1(x)\|_2 \ll \lambda^{1/p - 1/2}\, \|T(x)\|_p$ and $\|T_2(x)\|_q \ll \lambda^{1/p - 1/q}\, \|T(x)\|_p$. To do this we take the λ-cut of $T(x)$, i.e., $f_1(x) = \lambda$ when $|T(x)| > \lambda$, $f_1 = T(x)$ elsewhere, and $f_2(x) = f_1(x) - T(x)$. We obtain a decomposition $T(x) = f_1 + f_2$ with the functions f_1 and f_2 having the needed properties. Then we apply the operator of "step-hyperbolic" partial sums to both parts of this equality (see, for example, [26]). By the Marcinkiewicz multiplier theorem this operator is bounded in L_p, $1 < p < \infty$ and we have the desired decomposition (see [3], for details).

Now the interpolation property of entropy numbers (cf. [19, Sect. 12.1.12]) is applicable, and we get the following inequality

$$\varepsilon_m(B_p(j); L_q) \leqslant (\varepsilon_m(B_2(j); L_q))^{1-\theta}\, \varepsilon_1(B_q(j); L_q)^{\theta} \leqslant 2(\varepsilon_m(B_2(j); L_q))^{1-\theta}.$$

Again use the method used for the proof Theorem 1.1 with Lemma 3.3 replaced by this inequality.

Step 3. Let us consider the case $1 < p < q \leqslant 2$, $r_1 > 1/p - 1/q$. The Marcinkiewicz multiplier theorem yields that $B_{p'}(j)$ with $1/p + 1/p' = 1$ can be considered as the ball in the dual space of $B_p(j)$. Hence we can use Lemma 3.1 and as above we again use the method of Theorem 1.1 with Lemma 3.2, then proceed as in Steps 1 and 2.

Step 4. If $1 < p < 2 < q < \infty$, then the estimate follows from the previous estimates and the transitivity property of entropy numbers (Lemma 3.5).

Theorem 1.4 is proved. ▮

Remark 3.6. There exists another way to transfer estimates for the entropy numbers of a compact operator to the entropy numbers of its conjugate. To show this, we need the following result (it is a partial case of [30]; the method is not universal since the duality conjecture for entropy numbers has not been yet proved in full generality).

LEMMA 3.7. *Let* $U_{\mathbf{r}}$ *be the operator defined in Lemma* 3.5. *Then*

$$\varepsilon_{2m-1}(U_{\mathbf{r}}: L_p \to L_2) \leqslant [\varepsilon_m(U_{\mathbf{r}}: L_p \to L_2)]^{1/2} [\varepsilon_m(U_{\mathbf{r}}^*: L_2 \to L_{p'})]^{1/2}.$$

Now we have

$$\begin{aligned}
&\sup_{1 \leqslant k \leqslant m} (k^{-1} \log^{\nu-1} k)^{-r_1} \varepsilon_k(W_p^{\mathbf{r}}; L_2)\\
&\quad \leqslant \sup_{1 \leqslant k \leqslant m/2} (k^{-1} \log^{\nu-1} k)^{-r_1} \varepsilon_{2k-1}(W_p^{\mathbf{r}}; L_2)\\
&\quad \ll \Big(\sup_{1 \leqslant k \leqslant m} (k^{-1} \log^{\nu-1} k)^{-r_1} \varepsilon_k(W_p^{\mathbf{r}}; L_2)\Big)^{1/2}\\
&\qquad \times \Big(\sup_{1 \leqslant k \leqslant m} (k^{-1} \log^{\nu-1} k)^{-r_1} \varepsilon_k(W_2^{\mathbf{r}}; L_{p'})\Big)^{1/2}.
\end{aligned}$$

Comparing the left-hand and right-hand side estimates and using the inequality from Step 1 we obtain the estimate of Theorem 1.4 for $1 < p < 2$, $q = 2$, $r_1 > 1/p - 1/2$.

Proof of Theorem 1.5. The proof of this theorem repeats the proof of Theorem 1.4. The only alterations are the following estimates of polynomials of $B_p(f; j)$ (see [11], for $p = 2$, or [26], for $1 < p < \infty$):

For each $f \in H_p^{\mathbf{r}}$

$$\Big\| \sum_{r_1 j \leqslant (\mathbf{s}, \mathbf{r}) < r_1(j+1)} \delta_{\mathbf{s}}(f; \mathbf{x}) \Big\|_2 \ll 2^{-jr_1} j^{(\nu-1)/2}.$$

Proof of Theorem 1.2. The proof for the case $p = 2$, $r_1 > 1/2$ is the same as the proof for the space $W_2^{\mathbf{r}}$. If $1 < p < 2$ we can use Lemma 3.5 and Theorem 1.4 taking into account that the operator of convolution with the kernel $K_{\mathbf{r}'}$ takes $H_p^{\mathbf{r}}$ into $H_p^{\mathbf{r}+r'}$ (see [26]).

Remark 3.8. With the general result of [16] estimates of entropy numbers can also be derived from the corresponding estimates of widths (see, for example, [6, 7, 8, 26, 27]).

APPENDIX

Here we answer a question of Kuelbs and Li [15]. They considered the unit ball K_α, with $0 < \alpha < 2$, defined as

$$K_\alpha = \left\{ f(t) = T_\alpha g(t): 0 \leqslant t \leqslant 1, \int_{\mathbb{R}} g^2(u)\, du \leqslant 1 \right\},$$

where

$$T_\alpha g(t) = \int_0^t (t-u)^{(\alpha-1)/2} g(u)\, du \\ + \int_{-\infty}^0 ((t-u)^{(\alpha-1)/2} - (-u)^{(\alpha-1)/2})\, g(u)\, du.$$

They derived the order of ε-entropy from the estimates of the Gaussian measure: $\mathscr{H}_\varepsilon(K_\alpha, C) \equiv \varepsilon^{-2/(\alpha+1)}$ and asked whether a direct proof exists. Here we sketch a direct proof.

A standard argument (see, for example, [22, Sect. 14]) gives the embedding $K_\alpha \subset H_2^{(\alpha+1)/2}$. Therefore [9]

$$\mathscr{H}_\varepsilon(K_\alpha, C) \ll \varepsilon^{-2/(\alpha+1)}.$$

To prove the estimate from below it suffices to build in K_α a set of ε-distinguishable points. Let us take the function

$$\chi_k(x) = \begin{cases} 1, & x \in \left(\dfrac{2k-1}{2n+1}, \dfrac{2k}{2n+1}\right) \\ -1, & x \in \left(\dfrac{2k}{2n+1}, \dfrac{2k+1}{2n+1}\right) \\ 0, & \text{elsewhere} \end{cases}$$

and consider in K_α the set of functions $\{T_\alpha g_\varepsilon(x)\}$ where $g_\varepsilon(t) = \sum_{k=1}^n \varepsilon_k \chi_k(x)$, $\varepsilon_k \in \{+1, -1\}$, are all possible sums with $\varepsilon = +1$ or -1. Since the number of such functions is 2^n and the distance between each two of them $\gg n^{-2/(\alpha+1)}$ we get the estimate from below

$$\mathscr{H}_\varepsilon(K_\alpha, C) \gg \varepsilon^{-2/(\alpha+1)}. \quad \blacksquare$$

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